

# A refinement of theorems on vertex-disjoint chorded cycles

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## Abstract

In 1963, Corrádi and Hajnal settled a conjecture of Erdős by proving that, for all  $k \geq 1$ , any graph  $G$  with  $|G| \geq 3k$  and minimum degree at least  $2k$  contains  $k$  vertex-disjoint cycles. In 2008, Finkel proved that for all  $k \geq 1$ , any graph  $G$  with  $|G| \geq 4k$  and minimum degree at least  $3k$  contains  $k$  vertex-disjoint chorded cycles. Finkel’s result was strengthened by Chiba, Fujita, Gao, and Li in 2010, who showed, among other results, that for all  $k \geq 1$ , any graph  $G$  with  $|G| \geq 4k$  and minimum Ore-degree at least  $6k - 1$  contains  $k$  vertex-disjoint cycles. We refine this result, characterizing the graphs  $G$  with  $|G| \geq 4k$  and minimum Ore-degree at least  $6k - 2$  that do not have  $k$  disjoint chorded cycles.

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## 1 Introduction

All graphs in this paper are simple, unless otherwise noted. Additionally, when referring to cycles in a graph, “disjoint” is always taken to mean “vertex-disjoint.” For a graph  $G$ , we use  $V(G)$  and  $E(G)$  to denote the vertices and edges, respectively, and for a vertex  $v$ , we will use  $v \in G$  to denote  $v \in V(G)$ . For a vertex  $v \in G$ , and for a subgraph  $H$  of  $G$  (where possibly  $H = G$ ), the neighborhood of  $v$  in  $H$  is denoted by  $N_H(v)$ . The number of neighbors of  $v$  in  $H$  (i.e.,  $|N_H(v)|$ ) will be written by  $d_H(v)$ . Furthermore, we write  $|G|$  for the order of a graph  $G$ ,  $\overline{G}$  for its complement,  $\delta(G)$  for its minimum degree, and  $\alpha(G)$  for its independence number.

The minimum Ore-degree of a non-complete graph  $G$  is written  $\sigma_2(G)$ , and defined as

$$\sigma_2(G) := \min\{d_G(x) + d_G(y) : xy \in E(\overline{G})\};$$

that is,  $\sigma_2(G)$  is the minimum degree-sum of nonadjacent vertices.  $K_n$  is the complete graph on  $n$  vertices, and  $K_{s_1, \dots, s_t}$  is the complete  $t$ -partite graph with parts of size  $s_1, \dots, s_t$ . For graphs  $G$  and  $H$ ,  $G + H$  is the disjoint union of  $G$  and  $H$ , and  $G \vee H$  is the join of  $G$  and  $H$ .

In 1963, Corrádi and Hajnal verified a conjecture of Erdős, proving the following.

**CHT** **Theorem 1** (Corrádi-Hajnal, [3]). *Every graph  $G$  on  $|G| \geq 3k$  vertices with  $\delta(G) \geq 2k$  contains  $k$  disjoint cycles.*

This result of Corrádi and Hajnal has been generalized in various ways. One such generalization is a strengthening by Enomoto and Wang, who independently proved the following.

**EW** **Theorem 2** (Enomoto [5], Wang [14]). *Every graph  $G$  on  $|G| \geq 3k$  vertices with  $\sigma_2(G) \geq 4k - 1$  contains  $k$  disjoint cycles.*

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Both Theorems 1 and 2 are sharp, leading to the following natural question of Dirac.

**DiracQ** **Question 3** (Dirac, [4]). Which  $(2k - 1)$ -connected multigraphs do not contain  $k$  disjoint cycles?

Question 3 was answered in the case of simple graphs in [10], and then in multigraphs in [11]. Indeed, [10] together with [12] answer a more general question for simple graphs, describing graphs with minimum Ore-degree at least  $4k - 3$  with no  $k$  disjoint cycles. To avoid going into too many technical details, we only provide part of this description below.

**KKYT** **Theorem 4** ([10], [12]). *Given an integer  $k \geq 4$ , let  $G$  be a graph on  $|G| \geq 3k$  vertices with  $\sigma_2(G) \geq 4k - 3$ . Then  $G$  contains  $k$  disjoint cycles if and only if none of the following hold:*

1.  $\alpha(G) \geq |G| - 2k + 1$ .
2.  $G = (K_c + K_{2k-c}) \vee \overline{K_k}$  for some odd  $c$
3.  $G = (K_1 + K_{2k}) \vee \overline{K_{k-1}}$
4.  $|G| = 3k$  and  $\overline{G}$  is not  $k$ -colorable

In 2008 Finkel proved the following chorded-cycle analogue to Theorem 1.

**FinkelT** **Theorem 5** (Finkel, [7]). *Every graph  $G$  on  $|G| \geq 4k$  vertices with  $\delta(G) \geq 3k$  contains  $k$  disjoint chorded cycles.*

A stronger version of Theorem 5 was conjectured by Bialostocki, Finkel, and Gyárfás in [1], and proved by Chiba, Fujita, Gao, and Li in [2].

**CFGLT** **Theorem 6** (Chiba-Fujita-Gao-Li, [2]). *Let  $r$  and  $k$  be integers with  $r + k \geq 1$ . Every graph  $G$  on  $|G| \geq 3r + 4k$  vertices with  $\sigma_2(G) \geq 4r + 6k - 1$  contains a collection of  $r + k$  disjoint cycles such that  $k$  of these cycles are chorded.*

In particular, the following corollary holds.

**CFGLC** **Corollary 7** (Chiba-Fujita-Gao-Li, [2]). *Every graph  $G$  on  $|G| \geq 4k$  vertices with  $\sigma_2(G) \geq 6k - 1$  contains a collection of  $k$  disjoint chorded cycles.*

All hypotheses in Theorem 5 and Corollary 7 are sharp. First, since any chorded cycle contains at least four vertices, if  $|G| < 4k$  then  $G$  does not contain  $k$  disjoint chorded cycles. Second, the conditions  $\delta(G) \geq 3k$  and  $\sigma_2(G) \geq 6k - 1$  are best possible, as demonstrated by the two graphs below.

**G1** **Definition 8.** For  $n \geq 6k - 2$ , define  $G_1(n, k) := K_{3k-1, n-3k+1}$  (Figure 1a). For  $k \geq 2$ , define  $G_2(k) := K_{3k-2, 3k-2, 1}$  (Figure 1b).

For  $n \geq 6k - 2$ ,  $|G_1(n, k)| = n \geq 4k$  and  $\sigma_2(G_1(n, k)) = 6k - 2$ . Each chorded cycle in  $G_1(n, k)$  uses at least three vertices from each part, so  $G_1(n, k)$  does not contain  $k$  disjoint chorded cycles. For  $k \geq 2$ ,  $|G_2(k)| = 6k - 3 \geq 4k$  and  $\sigma_2(G_2(k)) = 6k - 2$ . Each chorded cycle in  $G_2(k)$  uses three vertices from each of the big parts, or the dominating vertex and at least two vertices from a big part, so  $G_2(k)$  does not contain  $k$  chorded cycles.

We can now ask a question similar to Question 3: which graphs  $G$  with  $\sigma_2(G) \geq 6k - 2$  do *not* contain  $k$  disjoint chorded cycles? Our main result is the following.

**main** **Theorem 9.** *For  $k \geq 2$ , let  $G$  be a graph with  $n := |G| \geq 4k$  and  $\sigma_2(G) \geq 6k - 2$ .  $G$  does not contain  $k$  disjoint chorded cycles if and only if  $G \in \{G_1(n, k), G_2(k)\}$ .*

The condition  $k \geq 2$  in Theorem 9 is necessary, as subdividing every edge of a graph results in a new graph with no chorded cycles. Thus, for  $k = 1$ , we obtain the following characterization, which is analogous to the characterization of acyclic graphs as the graphs for which there exists at most one path between every pair of vertices.

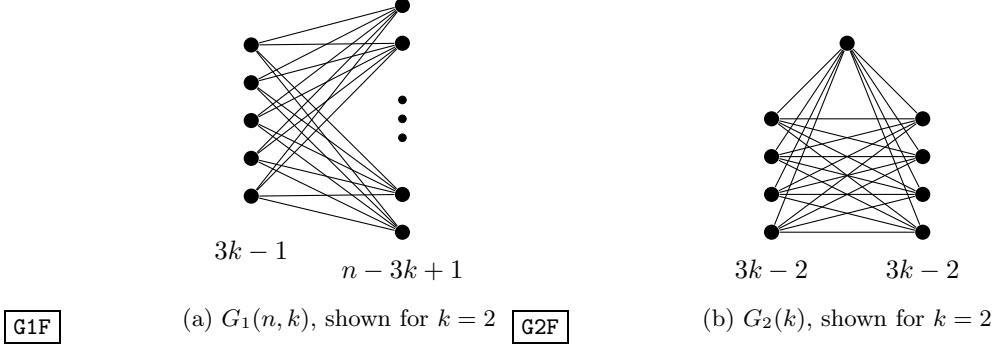


Figure 1: Graphs  $G_1(n, k)$  and  $G_2(k)$  from Definition 8.

**Proposition 10.** *A graph  $G$  has no chorded cycle if and only if for all  $uv \in E(G)$ ,  $G - uv$  has at most one path between  $u$  and  $v$ .*

Every graph  $G$  with  $\delta(G) \geq 3k - 1$  also satisfies  $\sigma_2(G) \geq 6k - 2$ . Therefore, Theorem 9 is a refinement of both Theorem 5 and Corollary 7. Two other immediate corollaries of Theorem 9 are listed here.

indep cor

**Corollary 11.** *For  $k \geq 2$ , let  $G$  be a graph with  $|G| \geq 4k$ ,  $\sigma_2(G) \geq 6k - 2$ , and  $\alpha(G) \leq n - 3k$ . Then  $G$  contains  $k$  disjoint chorded cycles.*

Every graph  $G$  with  $\sigma_2(G) \geq 6k - 2$  also satisfies  $\alpha(G) \leq n - 3k + 1$ . So, requiring  $\alpha(G) \leq n - 3k$  in Corollary 11 is equivalent to requiring the seemingly weaker condition  $\alpha(G) \neq n - 3k + 1$ .

**Corollary 12.** *For  $k \geq 2$ , let  $G$  be a graph with  $4k \leq |G| \leq 6k - 4$  and  $\sigma_2(G) \geq 6k - 2$ . Then  $G$  contains  $k$  disjoint chorded cycles.*

## 1.1 Outline

We present our result as follows. In Section 2, we detail the setup of our proof and present several important lemmas that will be used throughout our paper. In particular, we find and choose an ‘optimal’ collection of  $k - 1$  disjoint cycles, and use  $R$  to denote the subgraph induced by the vertices outside our collection. Then, in Section 3, we consider the case when  $R$  does not have a spanning path, and, in Section 4, we consider the case when  $R$  has a spanning path. We conclude our paper in Section 5 with some remarks on further extensions.

## 2 Setup and Preliminaries

setup

### 2.1 Notation

Let  $G$  be a graph, and let  $A, B \subseteq V(G)$ , not necessarily disjoint. We define  $\|A, B\| := \sum_{a \in A} |N_G(a) \cap B|$ .

When  $A = \{a\}$  or  $A$  is the vertex set of some subgraph  $\mathcal{A}$ , we will often replace  $A$  in the above notation with  $a$  or  $\mathcal{A}$ , respectively. Additionally, if  $\mathcal{L}$  is a collection of graphs, then  $\|A, \mathcal{L}\| = \|A, \bigcup_{L \in \mathcal{L}} V(L)\|$ . If  $A$  is

the vertex set of some subgraph  $\mathcal{A}$ , we will write  $G[\mathcal{A}]$  for  $G[A]$ , the subgraph of  $G$  induced by the vertices of  $\mathcal{A}$ . Furthermore, if  $\mathcal{B}$  is a subgraph of  $G$  with vertex set  $B$ , we will use  $\mathcal{A} \setminus \mathcal{B}$  to denote  $G[A \setminus B]$ , and if  $B = \{b_1, \dots, b_k\}$  and  $k$  is small, we will also use  $\mathcal{A} - b_1 - \dots - b_k$ . For a vertex  $v$ , we additionally write  $\mathcal{A} + v$  for  $G[A \cup \{v\}]$ .

If  $P = v_1 \dots v_m$  is a path, then for  $1 \leq i \leq j \leq m$ ,  $v_i P v_j$  is the path  $v_i \dots v_j$ . An  $n$ -cycle is a cycle with  $n$  vertices. A singly chorded cycle is a cycle with precisely one chord, and a doubly chorded cycle is a cycle with at least two chords.

## 2.2 Setup

We let  $k \geq 2$  and consider a graph  $G'$  on  $n$  vertices such that  $n \geq 4k$  and  $\sigma_2(G') = 6k - 2$ , where  $G'$  does not contain  $k$  disjoint chorded cycles. We then let  $G$  be a graph with vertex set  $V(G')$  such that  $E(G') \subseteq E(G)$  and  $G$  is “edge-maximal” in the sense that, for any  $e \in E(\overline{G})$ ,  $G + e$  does contain  $k$  disjoint chorded cycles. We then prove that  $G$  is  $G_1(n, k)$  or  $G_2(k)$ , which implies that  $G = G'$ , because any proper spanning subgraph of  $G_1(n, k)$  or  $G_2(k)$  has minimum Ore-degree less than  $6k - 2$ . Since we have already observed that  $G_1(n, k)$  and  $G_2(k)$  do not contain  $k$  disjoint chorded cycles, this will prove Theorem 9.

Note that  $G \not\cong K_n$ , else  $G$  contains  $k$  disjoint chorded cycles. So there exists  $e \in E(\overline{G})$ , and by our edge-maximality condition,  $G$  contains  $k - 1$  disjoint chorded cycles. Over all possible collections of  $k - 1$  disjoint chorded cycles in  $G$ , let  $\mathcal{C}$  be such a collection which satisfies the following conditions when  $R := G \setminus \mathcal{C}$ :

- (O1) the number of vertices in  $\mathcal{C}$  is minimum,
- (O2) subject to (O1), the total number of chords in the cycles of  $\mathcal{C}$  is maximum, and
- (O3) subject to (O1) and (O2), the length of the longest path in  $R$  is maximum.

We use the convention that  $P$  is a longest path in  $R$ . Since  $G[P]$  may have several paths spanning  $V(P)$  and the endpoints of such paths will behave in a similar manner, we let

$$\mathcal{P} := \{v \in V(P) : v \text{ is an endpoint of a path spanning } V(P)\}.$$

## 2.3 Preliminary Results

We begin with a number of observations about  $G$  that follow directly from our setup. In the interest of readability, the observations in this paragraph will be used in the text without citation. Since  $G$  does not contain  $k$  disjoint chorded cycles,  $R$  does not contain any chorded cycle, and for any  $C \in \mathcal{C}$ ,  $G[R \cup C]$  does not contain two disjoint chorded cycles. If  $p$  is an endpoint of  $P$  and has a neighbor in  $R \setminus P$ , we can extend  $P$ . Thus,  $\|p, R\| = \|p, P\|$ . If  $\|p, P\| \geq 3$ , then  $G[P]$  contains a chorded cycle, so  $\|p, R\| \leq 2$ . Similarly, to avoid a chorded cycle in  $R$ ,  $\|q, P\| \leq 3$  and for any  $v \in P$ ,  $\|v, P\| \leq 4$ . If  $p$  has two neighbors in  $P$ , then  $G[P]$  contains two distinct spanning paths.

An immediate corollary of (O1) is that, for any chorded cycle  $C \in \mathcal{C}$ , no vertex of  $C$  is incident to two chords; otherwise, we could replace  $C$  with a chorded cycle on fewer vertices. We will assume this fact in the proof of the following lemma.

RCedges

**Lemma 13.** *Let  $v \in R$  and  $C \in \mathcal{C}$ .*

4edgesL

- (1) *If  $\|v, C\| \geq 4$ , then  $\|v, C\| = 4 = |C|$ , and  $G[C] \cong K_4$ .*

3edgesL

- (2) *If  $\|v, C\| = 3$ , then  $|C| \in \{4, 5, 6\}$ . Moreover:*

3edgesL4

- (a) *if  $|C| = 4$ , then  $C$  has a chord incident to the non-neighbor of  $v$  (see Figure 2a);*

3edgesL5

- (b) *if  $|C| = 5$ , then  $C$  is singly chorded, and the endpoints of the chord are disjoint from the neighbors of  $v$  (see Figure 2b); and*

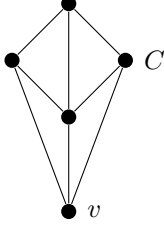
3edgesL6

- (c) *if  $|C| = 6$ , then  $C$  has three chords, with  $G[C] \cong K_{3,3}$  and  $G[C + v] \cong K_{3,4}$  (see Figure 2c).*

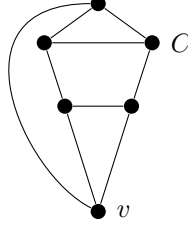
*Proof.* If there exist vertices  $c_1, c_2 \in C$  that are adjacent along the cycle of  $C$  such that  $\|v, C - c_1 - c_2\| \geq 3$ , then  $(C - c_1 - c_2) + v$  contains a chorded cycle with strictly fewer vertices than  $C$ , contradicting (O1). This proves that if  $\|v, C\| = 3$ , then  $|C| \leq 6$ . Similarly, if  $\|v, C\| \geq 4$ , then  $|C| = 4$  and  $\|v, C\| = 4$ . If  $\|v, C\| = 4$  and  $|C| = 4$ , then  $v$  together with a triangle in  $C$  give a doubly chorded 4-cycle, so by (O2),  $G[C] \cong K_4$ .

Suppose  $\|v, C\| = 3$ . If  $|C| = 4$ , then let  $c \in C$  be the non-neighbor of  $v$  in  $C$ . If  $c$  is not incident to a chord, then  $(C - c) + v$  gives a doubly chorded 4-cycle, preferable to  $C$  by (O2). This proves (a).

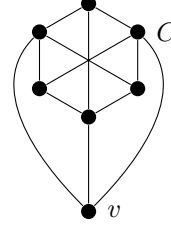
So  $|C| \in \{5, 6\}$ . Since the vertices in  $V(C) \setminus N_G(v)$  cannot be adjacent along the cycle  $C$ ,  $C - c + v$  contains a chorded cycle  $C'$  of the same length as  $C$ , for any  $c \in V(C) \setminus N_G(v)$ . If  $c$  is not incident to a



(a)  $|C| = 4$ ,  $\|v, C\| = 2$



(b)  $|C| = 5$ ,  $\|v, C\| = 2$



(c)  $|C| = 6$ ,  $\|v, C\| = 3$

Figure 2: Lemma 13(2)

chord, then  $C'$  has strictly more chords than  $C$ , violating (O2). So every vertex in  $V(C) \setminus N_G(v)$  is incident to a chord.

If  $|C| = 6$ , then  $v$  is adjacent to every other vertex along the cycle, and every  $c \in V(C) \setminus N_G(v)$  is incident to a chord. Since no vertex in  $C$  is incident to two chords, (O1) implies (c). If  $|C| = 5$ , then (O1) implies that the only possible chord has the two non-neighbors of  $v$  as its endpoints, which proves (b).  $\square$

C4bounds

**Lemma 14.** *Let  $Q$  be a path in  $R$  such that  $|Q| \geq 4$  and let  $C \in \mathcal{C}$ . If  $F \subseteq V(Q)$  such that  $|F| = 4$ , then  $\|F, C\| \leq 12$ . Furthermore, if  $G[C] \cong K_4$  and there exists an endpoint  $v$  of  $Q$  such that  $\|v, C\| \geq 3$ , then  $\|Q, C\| \leq 12$  with  $\|Q, C\| = 12$  only if  $\|v, C\| = 4$ .*

*Proof.* Assume  $\|F, C\| \geq 13$  for some  $F \subseteq V(Q)$ ,  $|F| = 4$ , and let  $u_1, u_2, u_3, u_4$  be the vertices of  $F$  in the order they appear on the path  $Q$ . By Lemma 13,  $G[C] \cong K_4$ , so there exists  $c \in C$  such that  $\|c, F\| \geq 4$ . Since  $\|\{u_1, u_4\}, C\| \geq 5$ , there exists  $i \in \{1, 4\}$  such that  $\|u_i, C\| \geq 3$ . So  $Q - u_i + c$  and  $C - c + u_i$  both contain chorded cycles, a contradiction.

To prove the second statement, suppose  $G[C] \cong K_4$  and let  $v$  be an endpoint of  $Q$  such that  $\|v, C\| \geq 3$ . Note that for every  $c \in C$ ,  $C - c + v$  and  $Q - v + c$  both contain chorded cycles if  $\|c, Q - v\| \geq 3$ . Thus,  $\|Q, C\| \leq 12$ , and furthermore, if  $\|Q, C\| = 12$ , then  $\|c, Q\| = 3$  and  $\|c, v\| = 1$  for every  $c \in C$ .  $\square$

noC5

**Lemma 15.** *If  $C \in \mathcal{C}$  and  $\|v_1, C\|, \|v_2, C\| \geq 3$  for distinct  $v_1, v_2 \in R$ , then  $|C| \in \{4, 6\}$ .*

*Proof.* If  $C \notin \{4, 6\}$ , then  $|C| = 5$  and  $N_C(v_1) = N_C(v_2)$ , by Lemma 13. Furthermore, Lemma 13, implies that there are two adjacent vertices  $c, c' \in N_C(v_1) = N_C(v_2)$ , but then  $v_1 c v_2 c' v_1$  is a chorded cycle contradicting (O1).  $\square$

In the following sections, we will often show that every  $C$  in  $\mathcal{C}$  is a 6-cycle. Furthermore, it will often be the case that there exists some  $u \in R$  such that  $\|u, C\| = 3$  for every  $C \in \mathcal{C}$ . The following lemma will be useful in considering the neighbors of  $u$  in  $R$  and their adjacencies in  $C$ .

uvnhd

**Lemma 16.** *Let  $C \in \mathcal{C}$  with  $|C| = 6$ , and let  $u, v \in R$  such that  $uv \in E(G)$ . If  $\|u, C\| = 3$  and  $\|v, C\| \geq 1$ , then  $N_C(u) \cap N_C(v) = \emptyset$ .*

*Proof.* By Lemma 13, we may assume that  $A = \{a_1, a_2, a_3\}$  and  $B = \{b_1, b_2, b_3\}$  are the partite sets of  $G[C] \cong K_{3,3}$  with  $N_C(u) = A$ . Suppose on the contrary that  $va_1 \in E(G)$ . Then  $ua_2 b_1 a_1 v u$  is a 5-cycle with chord  $ua_1$ . This contradicts (O1).  $\square$

P2P3

**Lemma 17.** *Suppose  $H$  is a graph with no chorded cycle. Let  $U$  and  $W$  be two disjoint paths in  $H$  and let  $u_1$  and  $u_2$  be the endpoints of  $U$ . Then  $\|\{u_1, u_2\}, W\| \leq 3$ . If equality holds, then  $u_1 \neq u_2$  and for some  $i \in [2]$ ,  $\|u_i, W\| = 2$  and  $\|u_{3-i}, W\| = 1$ , with the neighbor of  $u_{3-i}$  strictly between the neighbors of  $u_i$  on  $W$ ; in addition,  $\|U, W\| = 3$ .*

*Proof.* Let  $W = w_1 w_2 \dots w_t$  for some  $t \geq 1$ .  $\|u_1, W\| \leq 2$  and  $\|u_2, W\| \leq 2$ , as  $H$  does not contain a chorded cycle. Thus, if  $\|\{u_1, u_2\}, W\| \geq 3$ , we may assume that  $u_1 \neq u_2$ , and, without loss of generality,

that  $\|u_1, W\| = 2$  and  $\|u_2, W\| \geq 1$ . Suppose  $u_1w_i, u_1w_j \in E(H)$  such that  $i < j$ , and let  $u_2w_\ell \in E(H)$  for some  $\ell$ .

If  $\ell \leq i$ , then  $w_\ell W w_j u_1 U u_2 w_\ell$  is a cycle with chord  $u_1w_i$ . If  $\ell \geq j$ , then  $w_i W w_\ell u_2 U u_1 w_i$  is a cycle with chord  $u_1w_j$ . Thus, the neighbors of  $u_2$  in  $W$  are internal vertices of the path  $w_i W w_j$ . If  $\|u_2, W\| = 2$ , then suppose  $\ell$  is the largest index such that  $u_2w_\ell \in E(H)$ . However,  $w_i W w_\ell u_2 U u_1 w_i$  is a cycle containing a chord incident to  $u_2$ . So  $\|u_2, W\| = 1$ .

Now if  $v$  is an internal vertex on  $U$  such that  $vw_m \in E(H)$ , then by replacing  $u_2$  with  $v$ , we deduce that  $i \leq m \leq j$ . If  $m \leq \ell$ , then  $w_i W w_\ell u_2 U u_1 w_i$  is a cycle with chord  $vw_m$ , and if  $m > \ell$ , then  $w_\ell W w_j u_1 U u_2 w_j$  is a cycle with chord  $vw_m$ . This proves the lemma.  $\square$

### 3 Suppose $V(R) \neq V(P)$ .

R ne P

In this section, we make the assumption that  $V(R) \neq V(P)$ . That is, there exists some vertex  $v \in R \setminus P$ . In addition, we will use the convention that  $p$  and  $p'$  are the endpoints of  $P$ , and  $q$  (resp.  $q'$ ) is the neighbor of  $p$  (resp.  $p'$ ) on  $P$ . By the maximality of  $P$ ,  $vp \notin E(G)$  so that  $d_G(v) + d_G(p) \geq 6k - 2$ . Similarly for  $v$  and  $p'$ .

Our aim is to show that  $G = G_1(n, k)$ , which is a complete bipartite graph. To aid us, we define a set of vertices  $T := \{v \in R : d_R(v) = 2\}$ . We will show that  $T$  is contained in one of the partite sets of  $G_1(n, k)$ .

vp

**Lemma 18.** *If  $v \in R \setminus P$ , then  $\|\{v, p\}, C\| \leq 6$  for every  $C \in \mathcal{C}$ , with equality only if*

- (i)  $|C| \in \{4, 6\}$  and  $N_C(v) = N_C(p)$ , or
- (ii)  $\|p, C\| = |C| = 4$ .

*Proof.* Suppose  $v \in R \setminus P$  and  $\|\{v, p\}, C\| \geq 6$  for some  $C \in \mathcal{C}$ . If  $\|\{v, p\}, C\| \geq 7$ , then either  $\|v, C\| = 4$  or  $\|p, C\| = 4$ , so that  $G[C] \cong K_4$  by Lemma 13. If  $\|v, C\| = 4$ , then  $\|p, C\| = 0$ , lest we extend  $P$  by adding a neighbor of  $p$  in  $C$ , and replace said neighbor in  $C$  with  $v$ , violating (O3). If  $\|p, C\| = 4$ , then  $\|v, C\| \leq 2$ , else there exists  $c \in C$  such that  $C - c + v \cong K_4$ , and we can extend  $P$  by adding  $c$ , violating (O3). So,  $\|\{v, p\}, C\| \leq 6$ , and if equality holds, then either (ii) occurs, or  $\|v, C\| = \|p, C\| = 3$ . We may assume  $\|v, C\| = \|p, C\| = 3$ , so that  $|C| \in \{4, 5, 6\}$  by Lemma 13.

By Lemma 15,  $|C| \in \{4, 6\}$ . Suppose  $|C| = 4$  and  $\|v, C\| = \|p, C\| = 3$ . Note that  $G[N_C(v) \cup \{v\}]$  forms a chorded 4-cycle with at least the same number of chords as  $C$ . If  $p$  is adjacent to the vertex in  $V(C) \setminus N_G(v)$ , we use that vertex to extend  $P$ , violating (O3). So (i) holds.

Finally, suppose  $|C| = 6$ . By Lemma 13, if  $v$  and  $p$  do not have the same neighborhood, they are adjacent to disjoint sets of vertices, and  $C + p$  and  $C + v$  both contain  $K_{3,4}$ . In this case, we extend  $P$  using any  $c \in N_C(p)$ , and replace  $C$  with a chorded cycle in  $C - c + v$ . This violates (O3), so (i) holds.  $\square$

pbuds

**Lemma 19.** *For any  $v \in R \setminus P$ ,  $\|\{v, p\}, R\| \geq 4$ , so that  $\|v, R\| \geq 2$ . Moreover,  $|P| \geq 3$ .*

*Proof.* Let  $v \in R \setminus P$ . By the maximality of  $P$ ,  $pv \notin E(G)$ . Thus, by Lemma 18,

$$2(3k - 1) \leq d_G(v) + d_G(p) = \|\{v, p\}, \mathcal{C}\| + \|\{v, p\}, R\| \leq 6(k - 1) + \|\{v, p\}, R\|,$$

so  $\|\{v, p\}, R\| \geq 4$ . Since  $\|p, R\| \leq 2$ , it follows that  $\|v, R\| \geq 2$ . Then  $v$  and two of its neighbors form a path of length three in  $R$ , hence  $|P| \geq 3$ .  $\square$

R-Pv1v2

**Lemma 20.** *For any maximal path  $P'$  in  $R \setminus P$ , label the (not necessarily distinct) endpoints  $v_1$  and  $v_2$  so that  $\|v_1, P\| \leq \|v_2, P\|$ . Then:*

- (a)  $\|v_2, P\| \leq 2$ , and if  $v_1 \neq v_2$  then  $\|v_1, P\| \leq 1$ ,
- (b)  $d_R(v_1) = 2$  (this implies  $v_1 \in T \setminus V(P)$  so that  $T \setminus V(P) \neq \emptyset$ ), and
- (c) if  $\|v_2, P\| = 2$  and  $\|v_1, P\| = 1$ , then  $\|P' - v_1 - v_2, P\| = 0$ .

comp T

*Proof.* Since  $R$  contains no chorded cycle, no vertex in  $R \setminus P$  has three neighbors in  $P$ , so  $\|v_2, P\| \leq 2$ . Lemma 17 then gives (a) and (c).

It remains to show (b). If  $\|v_1, P\| = 0$ , then using Lemma 19 and the maximality of  $P'$ ,  $d_R(v_1) = \|v_1, P'\| = 2$ . If  $v_1 = v_2$ , then  $\|v_1, R\| = \|v_2, P\| = 2$ . So suppose  $\|v_1, P\| = 1$  and  $v_1 \neq v_2$ . Since  $\|v_2, P\| \geq \|v_1, P\| = 1$ , there exist  $a_1, a_2 \in P$  (perhaps  $a_1 = a_2$ ) such that  $v_1 a_1, v_2 a_2 \in E(G)$ . Then  $v_1 P' v_2 a_2 P a_1 v_1$  is a cycle. Since it has no chord,  $\|v_1, P'\| = 1$ , so  $\|v_1, R\| = 2$  and  $v_1 \in T$ .  $\square$

**pvnhd**

**Lemma 21.**  $d_R(p) = d_R(p') = 2$ . Additionally, for every  $v \in T \setminus V(P)$  and every  $C \in \mathcal{C}$ :

**pvnhdd**

(a)  $|C| \in \{4, 6\}$ ,

**pvnhdb**

(b)  $\|p, C\| = 3$ , and

**pvnhdc**

(c)  $N_C(v) = N_C(p)$ .

*Proof.* By Lemma 20,  $v \in T \setminus V(P)$  exists so that  $d_R(v) = 2$ . Lemma 19 implies  $\|\{v, p\}, R\| \geq 4$ , and hence,  $d_R(p) = 2$  and  $\|\{v, p\}, R\| = 4$ . Since  $vp \notin E(G)$ ,  $\|\{v, p\}, \mathcal{C}\| \geq (6k - 2) - 4 = 6(k - 1)$ . By Lemma 18,  $\|\{v, p\}, \mathcal{C}\| = 6$  for all  $C \in \mathcal{C}$ . If we can show that  $\|p, C\| = 3$  for all  $C \in \mathcal{C}$ , then we are done by Lemma 18.

If not, then there exists  $C \in \mathcal{C}$  such that  $\|p, C\| > 3$ , so  $\|p, C\| = 4$  and  $G[C] \cong K_4$  by Lemma 13. Thus,  $\|v, C\| = 2$ , and by Lemma 18, there exists  $u \in N_C(p')$ . Since  $\|p, P\| = 2$ ,  $P + u$  forms a chorded cycle, so since  $C - u + v$  also forms chorded cycles, we have a contradiction. Thus,  $\|p, C\| = 3$  as desired.  $\square$

From Lemma 21 we immediately obtain the following.

**mindeg**

**Corollary 22.**  $d_G(p) = d_G(p') = 3k - 1$ , and consequently,  $d_G(v) \geq 3k - 1$  for all  $v \in R \setminus P$ .

Recall that  $\mathcal{P}$  is the set of vertices in  $P$  that are the endpoint of a path spanning  $V(P)$ . Note Lemmas 18, 19, 20, and 21 apply to each  $p^* \in \mathcal{P}$ . Thus,  $\mathcal{P} \subseteq T$ , and furthermore, for all  $p_1^*, p_2^* \in \mathcal{P}$ ,  $N_C(p_1^*) = N_C(p_2^*)$ .

**|C|=6**

**Lemma 23.** For every  $C \in \mathcal{C}$ ,  $G[C] \cong K_{3,3}$ .

*Proof.* If not, by Lemma 13 and Lemma 21, we may assume that there exists  $C \in \mathcal{C}$  with  $|C| = 4$ . Suppose  $V(C) = \{c_1, c_2, c_3, c_4\}$ . Let  $v \in T \setminus V(P)$ , which we know exists by Lemma 20. By Lemmas 13 and 21, we may assume that  $N_C(p) = N_C(p') = N_C(v) = \{c_1, c_2, c_3\}$  and  $c_2 c_4 \in E(G)$ . Since  $\|p, P\| = 2$  by Lemma 21,  $P + c_1$  and  $C - c_1 + v$  contain chorded cycles, a contradiction.  $\square$

For the remainder of this section, we will use the fact that for each  $C \in \mathcal{C}$ ,  $G[C] \cong K_{3,3}$  and, that there exist  $A \subseteq C$  such that  $A$  is a partite set of  $C$  and such that, for every  $p^* \in \mathcal{P}$ ,  $N_C(p^*) = A$ , without mentioning Lemmas 21, and 23.

**nhdofv**

**Lemma 24.** For every  $C \in \mathcal{C}$ , if  $v \in R \setminus P$  has a neighbor in  $C$ , then  $N_C(v) \subseteq N_C(p)$ , unless  $|N_C(v)| = 1$ .

*Proof.* Fix  $C \in \mathcal{C}$ , and let  $A = \{a_1, a_2, a_3\}$  and  $B = \{b_1, b_2, b_3\}$  be the partite sets of  $C$  such that  $N_C(p) = N_C(p') = A$ . Suppose on the contrary, there exists  $v \in R \setminus P$  with  $|N_C(v)| \geq 2$  such that, say  $vb_3 \in E(G)$ .

By Lemma 21,  $\|p, P\| = 2$  so that  $P + a_i$  contains a chorded cycle for each  $i \in [3]$ . If  $vb_2 \in E(G)$ , then  $vb_3 a_3 b_1 a_2 b_2 v$  is a cycle with chord  $a_2 b_3$ . However,  $P + a_1$  also contains a chorded cycle, a contradiction.

So we may assume that  $va_3 \in E(G)$ . However,  $vb_3 a_2 b_2 a_3 v$  is a 5-cycle with chord  $a_3 b_3$  contradicting (O1). Thus,  $N_C(v) \subseteq A = N_C(p)$ , as desired.  $\square$

**R-P=T**

**Lemma 25.**  $R \setminus P$  is an independent set, and  $V(R \setminus P) \subseteq T$ .

*Proof.* Suppose  $R \setminus P$  is not an independent set. Then there exists a maximal path  $P'$  in  $R \setminus P$  with distinct endpoints  $v_1$  and  $v_2$ , labeled as in Lemma 20. Thus,  $\|v_2, P\| \leq 2$ , and, hence,  $d_R(v_2) \leq 4$ . Since  $pv_2 \notin E(G)$ , Lemma 22 implies that  $d_G(v_2) \geq 3k - 1 > 4$ , which implies that there exists  $C \in \mathcal{C}$  such that  $v_2$  has a neighbor  $c \in C$ .

Let  $A = \{a_1, a_2, a_3\}$  and  $B = \{b_1, b_2, b_3\}$  be the partite sets of  $C$  such that  $N_C(p) = N_C(p') = A$ . By Lemmas 20 and 21,  $v_1 \in T \setminus V(P)$  and  $N_C(v_1) = A$ . We can assume  $a_1 \neq c$ , so that there exists a path  $W$

in  $C - a_1$  that contains  $a_2$  and  $a_3$  for which  $c$  is an endpoint. Since  $\|v_1, W\| \geq 2$  and  $v_2$  is adjacent to an endpoint of  $W$ ,  $\|\{v_1, v_2\}, W\| \geq 3$  and Lemma 17 implies there is a chorded cycle in  $G[V(P') \cup V(C - a_1)]$ . However, as  $\|p, P\| = 2$ ,  $P + a_1$  also contains a chorded cycle, a contradiction.  $\square$

Let  $\mathcal{S} := N_{\mathcal{C}}(p)$ , and let  $\mathcal{T} := ((\bigcup_{C \in \mathcal{C}} V(C)) \setminus \mathcal{S}) \cup T$ .

bipar\_sub

**Proposition 26.**  $G[\mathcal{S} \cup \mathcal{T}] \cong K_{3k-3, |\mathcal{T}|}$ , and no vertex in  $G$  has neighbors in both  $\mathcal{S}$  and  $\mathcal{T}$ .

*Proof.* By Lemma 23,  $\mathcal{C}$  consists of  $k - 1$  copies of  $K_{3,3}$ . Lemmas 21 and 25 tell us that, for every  $v \in R \setminus P$ ,  $N_{\mathcal{C}}(v) = \mathcal{S}$ . Given  $C \in \mathcal{C}$ ,  $a \in V(C) \cap \mathcal{T}$ , and  $v \in R \setminus P$ , we can create a chorded cycle  $C'$  by swapping  $a$  and  $v$  in  $C$ . Note  $G[C'] \cong K_{3,3}$ , and we have not changed any vertices in  $P$ . Then replacing  $C$  with  $C'$  in  $\mathcal{C}$  results in a collection of  $k - 1$  chorded cycles satisfying (O1) through (O3). Thus all the previous lemmas apply, and, in particular, Lemma 20 and Lemma 25 imply that  $a \in T$ . So by Lemma 21, and the fact that  $N_C(a) = V(C) \cap \mathcal{S}$ , we conclude  $N_{\mathcal{C}}(a) = \mathcal{S}$ . Hence, every vertex in  $\mathcal{T}$  is adjacent to every vertex in  $\mathcal{S}$ , and  $G[\mathcal{S} \cup \mathcal{T}]$  contains a copy of  $K_{|\mathcal{S}|, |\mathcal{T}|}$ .

We claim  $G[\mathcal{S} \cup \mathcal{T}]$  has no additional edges. Note  $|\mathcal{T}| > 3(k - 1)$  and  $|\mathcal{S}| = 3(k - 1)$ . If there exists any edge with both endpoints in  $\mathcal{T}$ , or both endpoints in  $\mathcal{S}$ , then we find a set of  $k - 1$  chorded cycles,  $k - 2$  of which are 6-cycles, and one of which is a 4-cycle, violating (O1). So  $G[\mathcal{S} \cup \mathcal{T}] \cong K_{|\mathcal{S}|, |\mathcal{T}|} \cong K_{3k-3, |\mathcal{T}|}$ .

If any vertex of  $V(G) \setminus (\mathcal{S} \cup \mathcal{T})$  has neighbors in both  $\mathcal{S}$  and  $\mathcal{T}$ , then in a similar manner, we find  $k - 1$  disjoint chorded cycles, one of which is a 5-cycle and the rest of which are 6-cycles, again violating (O1).  $\square$

Recall that  $q$  and  $q'$  were defined as the neighbors of  $p$  and  $p'$ , respectively, on  $P$ . Since  $\|p, P\| = 2$  by Lemma 21, there exists  $w \in N_R(p) \setminus \{q\}$ . As a consequence of Proposition 26,  $w \neq p'$ . Now the neighbor of  $w$  on  $pPw$  is the endpoint of a path that spans  $V(P)$ . Thus,  $|\mathcal{P}| \geq 3$ .

|P|=3

**Lemma 27.**  $|\mathcal{P}| = 3$

*Proof.* Suppose  $|\mathcal{P}| \geq 4$ , with  $p_1, p_2, p_3, p_4$  the first four members of  $\mathcal{P}$  along  $P$ . In particular,  $p_1 = p$ . Fix  $C \in \mathcal{C}$ , and let  $A = \{a_1, a_2, a_3\}$  and  $B = \{b_1, b_2, b_3\}$  be the partite sets of  $C$  such that  $N_C(p_i) = A$  for each  $i \in [4]$ .

By Lemma 16,  $N_C(q) \subseteq B$ . So in particular,  $q \neq p_2$ . If  $q$  has a neighbor in  $C$ , say  $b_1$ , then  $qb_1a_1b_2a_2p_1q$  is a 6-cycle with chord  $p_1a_1$  and  $p_2p_4a_3p_2$  is a cycle with chord  $p_3a_3$ , a contradiction.

So we may assume that for every  $C \in \mathcal{C}$ ,  $N_C(q) = \emptyset$ . That is,  $\|q, R\| = d_G(q)$ . Since  $\|p_3, P\| = 2$  by Lemma 21,  $q$  is not adjacent to  $p_3$ . Then since  $d_G(p_3) = 3k - 1$  by Corollary 22,  $d_G(q) \geq 3k - 1 \geq 5$ . Since  $\|q, P\| \leq 3$ ,  $q$  must be adjacent to two vertices  $v_1, v_2 \in R \setminus P$ . By Lemma 21,  $N_C(v_1) = N_C(v_2) = A$ . However, this yields the cycles  $v_1qv_2a_2b_1a_1v_1$  and  $p_2p_4a_3p_2$  with chords  $v_1a_2$  and  $p_3a_3$ , respectively, a contradiction.  $\square$

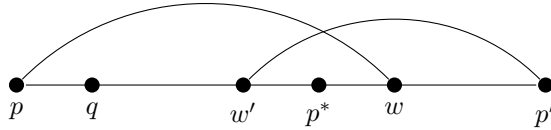


Figure 3: Setup for Lemma 28

pstar

K23

**Lemma 28.**  $G[P] \cong K_{2,3}$

*Proof.* By Lemma 27, we may assume that  $\mathcal{P} = \{p, p', p^*\}$ . Recall that  $\mathcal{P} \subseteq \mathcal{T}$ , so Lemma 26 implies that  $\mathcal{P}$  is an independent set. Lemma 21 implies that  $\|p, P\| = \|p', P\| = \|p^*, P\| = 2$ , so there exist  $w$  and  $w'$  on  $P$  such that  $w \neq q$  and  $w' \neq q'$  and  $N_P(p) = \{q, w\}$  and  $N_P(p') = \{q', w'\}$ . Furthermore, since  $|\mathcal{P}| = 3$ , and both the neighbor of  $w$  on  $pPw$  and the neighbor of  $w'$  on  $w'Pp'$  are in  $\mathcal{P}$ , we can conclude that  $w \neq w'$  and  $N_P(p^*) = \{w, w'\}$ , i.e.  $wPw'$  is the path on three vertices  $wp^*w'$ .



Since  $G[P]$  does not contain a chorded cycle,  $qq' \notin E$ , so if  $w = q'$  and  $w' = q$ , then  $G \cong K_{2,3}$ . So if  $G \not\cong K_{2,3}$ , then without loss of generality we can assume that  $q \neq w'$  as in Figure 3. Thus,  $qp', pw' \notin E(G)$  so, by Corollary 22,  $d_G(q), d_G(w') \geq 3k - 1$ .

Fix  $C \in \mathcal{C}$  with partite sets  $A = \{a_1, a_2, a_3\}$  and  $B = \{b_1, b_2, b_3\}$  such that  $N_C(p) = N_C(p') = N_C(p^*) = A$ . By Lemma 16,  $N_C(q) \subseteq B$  and  $N_C(w') \subseteq B$ .

Since  $d_G(q) \geq 3k - 1$ ,  $\|q, C \cup R\| \geq (3k - 1) - 3(k - 2) = 5$ . Also,  $\|q, P\| \leq 3$ . This holds for  $w'$  as well. Thus, both  $q$  and  $w'$  have two neighbors in  $B \cup (R \setminus P)$ . Let  $v_1$  and  $v_2$  be distinct vertices in  $B \cup (R \setminus P)$  such that  $v_1q, v_2w' \in E(G)$ . We may assume that  $v_2 \neq b_3$ . Observe that  $N_C(v_1) = N_C(v_2) = A$ . Then the cycle  $pqv_1a_1b_3a_3p$  has chord  $pa_1$ , and the cycle  $w'v_2a_2p'Pw'$  has chord  $a_2p^*$ , a contradiction.  $\square$

**Lemma 29.**  $G = G_1(n, k)$

*Proof.* By Lemma 28, let  $\{p_1, p_2, p_3\}$  and  $\{q_1, q_2\}$  denote the partite sets of  $G[P]$ . Recall that  $\mathcal{P} \subseteq T$  so that  $G[\mathcal{S} \cup \mathcal{T}]$  contains every vertex of  $G$  except for  $q_1$  and  $q_2$ .

By Lemmas 21 and 25 and Corollary 22,  $\|v, P\| = 2$  for all  $v \in R \setminus P$ , and by Proposition 26,  $N_R(v) = \{q_1, q_2\}$ . Since  $\mathcal{T}$  is an independent set in  $G$ , for each  $u \in \mathcal{T} \setminus T$ ,  $\|u, R\| \geq (3k - 1) - 3(k - 1) = 2$ . Thus,  $uq_1, uq_2 \in E(G)$ , and so  $N_G(q_i) \supseteq \mathcal{T}$  for  $i \in [2]$ . That is,  $G \supseteq K_{|\mathcal{S}|+2, |\mathcal{T}|} = K_{3k-1, |G|-3k+1} = G_1(n, k)$ . Since adding any edge to  $G_1(n, k)$  results in a graph with  $k$  disjoint chorded cycles, we conclude  $G = G_1(n, k)$ .  $\square$

## 4 Suppose $V(R) = V(P)$

long path

In this section, we assume  $V(P) = V(R)$ . Since adding any edge to  $G$  results in  $k$  chorded cycles, by (O1)  $|P| \geq 4$ . If  $|P| \geq 6$ , we label  $P = p_1q_1r_1 \cdots r_2q_2p_2$ . Note that, since  $G[R]$  has no chorded cycles, for every  $v \in R$ ,  $\|v, R\| \leq 4$ . When  $|P| = 5$ , we let  $P = p_1q_1rq_2p_2$ , and when  $|P| = 4$ , we let  $P = p_1q_1q_2p_2$ . We call an edge in  $E(G[P]) \setminus E(P)$  a *hop*. If  $Q = v_1 \cdots v_{|R|}$  is a spanning path of  $R$ , then we call an edge  $v_i v_j$  a *hop* (on  $Q$ ) if  $|i - j| > 1$ .

hops

**Lemma 30.** *If  $Q = v_1 \cdots v_{|R|}$  is a spanning path of  $R$  and  $v_i v_j$  is a hop with  $i < j$ , then  $v_{i+1}$  and  $v_{i+2}$  cannot both be incident to hops, and similarly,  $v_{j-1}$  and  $v_{j-2}$  cannot both be incident to hops.*

*Proof.* Suppose that, on the contrary,  $v_{i+1}v_k$  and  $v_{i+2}v_{k'}$  are both hops. Note that, if we consider only the hop  $v_i v_j$  and the hop  $v_{i+1}v_k$ ,  $v_j v_i v_{i+1} Q v_j$  is a chorded cycle if  $i + 3 \leq k \leq j$ , and  $v_k Q v_i v_j Q v_{i+1} v_k$  is a chorded cycle if  $k \leq i - 1$ , so  $k > j$ . Repeating this argument but now only considering the hops  $v_{i+1}v_k$  and  $v_{i+2}v_{k'}$  gives us that  $k' > k$ , but then  $v_i v_{i+1} v_{i+2} v_{k'} Q v_j v_i$  is a cycle with chord  $v_{i+1}v_k$ , a contradiction. By symmetry, the lemma holds.  $\square$

pchords

**Lemma 31.** *For any  $p \in \mathcal{P}$ ,  $d_R(p) = 2$  unless  $R$  is a path.*

*Proof.* Let  $v_1 \cdots v_{|R|}$  be a spanning path in  $R$ , and let  $p = v_1$ . Assume  $d_R(p) = 1$ , and that  $R$  is not a path. Since  $R$  is not a path, hops exist. Let  $v_i v_j$ ,  $i < j$ , be a hop such that for all  $k, j < k \leq |R|$ ,  $v_k$  is not incident to a hop. Note, because  $d_R(p) = 1$ , that  $i \neq 1$ .

Let  $D$  be the cycle  $v_j v_i v_{i+1} \cdots v_{j-1} v_j$ . Since  $R$  contains no chorded cycles,  $v_j$  is incident to exactly one hop and  $v_{j-1}$  is incident to at most one hop. If  $v_{j-1}$  is not incident to a hop let  $x = v_{j-1}$  and  $y = v_j$ , and if  $v_{j-1}$  is incident to exactly one hop, let  $x = v_{j-2}$  and  $y = v_{j-1}$ . By Lemma 30, when  $v_{j-1}$  is incident to a hop,  $v_{j-2}$  is not incident to a hop, so in either case,  $xy \in E(D)$ ,  $d_R(x) + d_R(y) \leq 5$ , and  $px, py \notin E(G)$ . Therefore,

$$2\|p, C\| + \|\{x, y\}, C\| \geq 2(6k - 2) - (2\|p, R\| + \|\{x, y\}, R\|) > 12(k - 1).$$

So there exists  $C \in \mathcal{C}$  such that  $2\|p, C\| + \|\{x, y\}, C\| \geq 13$ . Thus,  $\|v, C\| = 4$  for some  $v \in \{p, x, y\}$ , and by Lemma 13,  $G[C] \cong K_4$ . Further,  $\|\{x, y\}, C\| \geq 5$  so that there exists  $c \in C$  such that  $xc, yc \in E(G)$  and  $D + c$  contains a chorded cycle. Also  $2\|p, C\| \geq 5$ , which implies  $\|p, C - c\| \geq 2$  so that  $C - c + p$  contains a chorded cycle, a contradiction.  $\square$

**R6toC**

**Lemma 32.** *If  $|R| \geq 6$ , then there exists  $F^+ \subseteq V(R)$  such that  $|F^+| = 6$  and such that for every  $C \in \mathcal{C}$  and every pair of distinct vertices  $u, u' \in F^+$ ,  $\|\{u, u'\}, C\| \geq 1$ .*

*Proof.* First we find  $F^+ \subseteq V(R)$  such that  $\|F^+, R\| \leq 15$ . If  $R$  is a path, this is trivial, so we assume  $R$  has at least one hop. By Lemmas 30 and 31,  $p_i$  is incident to a hop so that  $q_i$  and  $r_i$  cannot both be incident to hops. If  $d_R(r_i) \leq 3$  for some  $i \in [2]$ , then since  $d_R(q_i) \leq 3$  and  $d_R(p_i) = 2$  by Lemma 31,  $\|\{p_i, q_i, r_i\}, R\| \leq 7$ . If  $d_R(r_i) = 4$ , then  $d_R(p_i) = d_R(q_i) = 2$ , so that  $\|\{p_i, q_i, r_i\}, R\| \leq 8$ . Therefore,  $F^+ := \{p_1, q_1, r_1, r_2, q_2, p_2\}$  suffices when either  $d_R(r_1) \leq 3$  or  $d_R(r_2) \leq 3$ . In this case, we let  $r_1^* = r_1$ .

When  $d_R(r_1) = d_R(r_2) = 4$ ,  $|R| \geq 7$ , since  $R$  has no chorded cycles, and there exists a vertex  $u$  following  $r_1$  on  $P$  with  $d_R(u) \leq 3$ . Here, we let  $F^+ := \{p_1, q_1, u, r_2, q_2, p_2\}$ . and let  $r_1^* = u$ . Thus, in both cases,  $F^+ = \{p_1, q_1, r_1^*, r_2, q_2, p_2\}$ .

We claim that we can partition  $F^+$  into three sets so that each set will consist of two nonadjacent vertices. Define  $F_1 := \{p_1, q_1, r_1^*\}$  and  $F_2 := \{p_2, q_2, r_2\}$ , and let  $H$  be the subgraph of  $G$  on the vertex set  $F^+$  containing precisely those edges of  $G$  with one endpoint in  $F_1$  and the other in  $F_2$ . Because  $R$  contains no chorded cycle, every vertex in  $F_2$  has at most two neighbors in  $F_1$ , and vice-versa. That is,  $H \subseteq 3K_2$ . Therefore we can label  $F_1 = \{f_1, f_2, f_3\}$  so that  $f_1p_2, f_2q_2$ , and  $f_3r_2$  are all nonedges.

Therefore,  $\|F^+, C\| \geq 3(6k - 2) - 15 = 18(k - 1) - 3$ . Suppose there exists  $C \in \mathcal{C}$  for which  $\|F^+, C\| \leq 14$  so that there exists  $C' \in \mathcal{C}$  such that  $\|F^+, C'\| \geq 19$ . If we can find  $v_1, v_2 \in F^+$  such that  $\|\{v_1, v_2\}, C'\| \leq 6$ , then  $\|F' - v_1 - v_2, C'\| \geq 13$ , contradicting Lemma 14. So for  $F^+ = \{v_1, v_2, \dots, v_6\}$ ,  $\|\{v_i, v_{i+1}\}, C'\| \geq 7$  for  $i \in \{1, 3, 5\}$ . However this implies  $\|\{v_1, v_2, v_3, v_4\}, C'\| \geq 14$ , a contradiction to Lemma 14.

Thus,  $\|F^+, C\| \geq 15$  for every  $C \in \mathcal{C}$ . If there exists a pair of distinct vertices  $u, u' \in F^+$  such that  $\|\{u, u'\}, C\| = 0$ , then  $\|F^+ - u - u', C\| \geq 15$ , again a violation of Lemma 14.  $\square$

**struct1**

**Lemma 33.** *There exists  $F \subseteq V(R)$  such that  $p_1, p_2 \in F$ ,  $|F| = 4$  and*

**Fdeg**

(a)  $\|F, C\| \geq 12(k - 1) - 2$  if  $R \cong K_{2,3}$ ,  $\|F, C\| \geq 12(k - 1) + 2$  if  $R$  is a path, and  $\|F, C\| \geq 12(k - 1)$  otherwise, and

**paths**

(b) if  $R$  is not a path, then for every  $u \in F$ , there exists a path  $Q$  in  $R - u$  such that  $F - u \subseteq V(Q)$ .

*Proof.* If  $R$  is a path or  $R \cong K_{2,3}$ , let  $F := \{p_1, q_1, q_2, p_2\}$ . When  $R$  is a path,  $\|F, R\| = 6$ , and  $p_1q_2, p_2q_1 \notin E(G)$ ; when  $R \cong K_{2,3}$ ,  $\|F, R\| = 10$ , and  $p_1p_2, q_1q_2 \notin E(G)$ . In both cases, (a) and (b) hold.

So we assume  $R \not\cong K_{2,3}$  and  $R$  is not a path. By Lemma 31, for  $i \in [2]$ ,  $\|p_i, P\| = 2$ . Thus,  $p_i$  has a neighbor  $w_i \in P - q_i$ . Let  $t_i$  denote the neighbor of  $w_i$  on  $w_i P p_i$ . Observe that  $t_i \in \mathcal{P}$ , so by Lemma 31,  $\|t_i, P\| = 2$ . Suppose  $t_1 \neq t_2$ , and, in this case, let  $F := \{p_1, t_1, t_2, p_2\}$ . Then  $F \subseteq \mathcal{P}$ , so (b) holds and  $\|F, R\| \leq 8$ . If either  $p_1t_1, p_2t_2 \notin E(G)$  or  $p_1t_2, p_2t_1 \notin E(G)$ , then (a) holds. Suppose (say)  $p_1t_1 \in E(G)$ . Then  $t_1 = q_1$ , and  $t_1p_2 \notin E(G)$ . Then  $w_2 \notin \{p_1, t_1\}$ , hence  $t_2 \notin \{t_1, w_1\} = N_R(p_1)$ , so also  $p_1t_2 \notin E(G)$ . So in this case also, (a) holds.

So assume  $t_1 = t_2$ , which implies  $\|u, P\| = 2$  for all  $u \in V(P) - w_1 - w_2$ , as otherwise  $R$  contains a chorded cycle. Also, when  $t_1 = t_2$ , we may assume that  $q_1 \neq w_2$  since  $R$  is not isomorphic to  $K_{2,3}$ . In this case, let  $F := \{p_1, q_1, t_1, p_2\}$  and note that  $p_1t_1, q_1p_2 \notin E(G)$ . Since  $d_R(u) = 2$  for all  $u \in F$ , (a) holds. Since  $t_1 = t_2$ ,  $p_1w_1t_1w_2p_2$  is a path in  $R - q_1$  containing  $F - q_1$  and  $F - q_1 \subseteq \mathcal{P}$ , (b) holds.  $\square$

**Corollary 34.**  *$R$  is not a path.*

*Proof.* Let  $F \subseteq V(R)$  be as guaranteed in Lemma 33. If  $R$  is a path, then  $\|F, C\| \geq 12(k - 1) + 2$ , so that there exists  $C \in \mathcal{C}$  such that  $\|F, C\| \geq 13$ , which violates Lemma 14. So  $R$  is not path.  $\square$

**struct2**

**Lemma 35.** *Let  $F \subseteq V(R)$  be as guaranteed in Lemma 33. If  $\|F, C\| = 12$  for any  $C \in \mathcal{C}$ , then  $G[C] \cong K_{3,3}$ .*

*Proof.* Let  $F \subseteq V(R)$  be as guaranteed in Lemma 33 and let  $C \in \mathcal{C}$ . Suppose that  $\|F, C\| = 12$ . By Lemmas 14 and 33, this is true for all  $C \in \mathcal{C}$ , unless  $R \cong K_{2,3}$ . By Lemmas 13 and 15,  $C \cong K_{3,3}$  unless  $|C| = 4$ , so assume  $|C| = 4$ . Note that for any  $u \in F$  and  $c \in C$ , if  $C - c + u$  is a chorded cycle, then  $\|c, F - u\| \leq 2$ , because there exists a path  $Q$  in  $R$  such that  $F - u \subseteq V(Q)$  and  $G[Q + c]$  cannot contain a chorded cycle.

First assume that  $C$  is singly chorded, so we can label  $V(C) = \{c_1, c_2, c_3, c_4\}$  such that  $c_1c_2c_3c_4$  is a cycle and  $c_2c_4$  is the chord. By Lemma 13,  $\|u, C\| = 3$  for every  $u \in F$ , and  $\|c_i, F\| = 4$ , for  $i \in \{1, 3\}$ . Recall that  $p_1, p_2 \in F$  so that  $C - c_1 + p_1$  and  $P - p_1 + c_1$  both contain chorded cycles, a contradiction.

So for the remainder of the proof, we assume  $G[C] \cong K_4$ , with  $V(C) = \{c_1, c_2, c_3, c_4\}$ . Fix  $u \in F$ , and by Lemma 33, let  $Q$  be a path in  $R - u$  such that  $F - u \subseteq V(Q)$ . Suppose  $\|u, C\| = 3$ , so  $\|F - u, C\| = 9$ , and there exists  $c \in C$  such that  $c$  is adjacent to all three vertices in  $F - u$ . This implies  $Q + c$  and  $C - c + u$  both contain chorded cycles, a contradiction.

Now suppose  $\|u, C\| = 2$  and  $N_C(u) = \{c_1, c_2\}$ . Then  $\|F - u, C\| = 10$ , and there exist two vertices in  $C$  adjacent to all three vertices in  $F - u$ . If  $c'$  is one of these two vertices and  $c' \notin \{c_1, c_2\}$ , then  $Q + c'$  and  $C - c' + u$  both contain chorded cycles, a contradiction. Therefore, every vertex in  $F$  is adjacent to both  $c_1$  and  $c_2$ . Since  $\|F, C\| = 12$  and  $\|u, C\| = 2$ , there exists  $v \in F - u$  such that  $\|v, C\| = 4$ . By Lemma 33, there exists a path  $Q'$  in  $R - v$  such that  $F - v \subseteq V(Q')$ , so that  $C - c_1 + v$  and  $Q' + c_1$  both contain chorded cycles, a contradiction.

So  $\|u, C\| \in \{0, 1, 4\}$ , for every  $u \in F$ . Since  $\|F, C\| = 12$ , there exists  $u' \in F$  such that  $\|u', C\| = 0$  and  $\|u, C\| = 4$  for every  $u \in F - u'$ . By Lemma 33,  $p_1, p_2 \in F$ , so we may assume  $\|p_1, C\| = 4$ . Thus, for all  $c \in C$ ,  $C - c + p_1$  is a chorded cycle, and further  $\|c, P - p_1\| \leq 2$ , else  $P - p_1 + c$  contains a chorded cycle. Therefore, if  $\|R \setminus F, C\| > 0$ , we can pick  $c$  such that  $\|c, P - p_1\| \geq 3$  so that  $P - p_1 + c$  has a chorded cycle, a contradiction.

Thus  $\|R \setminus F, C\| = 0$ . By Lemma 32,  $|R| \leq 5$ , as otherwise we can find  $F^+ \subseteq V(R)$  with  $|F^+| = 6$  so that for distinct  $v, v' \in F^+ \setminus F$ ,  $\|\{v, v'\}, C\| \geq 1$ , a contradiction. If  $|R| = 4$ , then  $u'$  has a neighbor  $v \in F - u'$ . Since  $R$  is not a path, by Lemma 31  $R \cong C_4$ , so replacing  $C$  with  $C' := C - c + v$  in  $\mathcal{C}$  gives a collection of  $k - 1$  chorded cycles that satisfies (O1) - (O3), but  $R' := R - v + c$  has a path  $P'$  such that  $|P'| = |R'|$  and such that  $u'$  is an endpoint and such that  $\|u', R'\| = 1$ . This is a contradiction to Lemma 31.

So assume  $|R| = 5$  so that  $P = p_1q_1r q_2p_2$ . By Lemma 31, either  $p_1r, p_2r \in E(G)$ , or  $R \in \{C_5, K_{2,3}\}$ . In each of these cases, we can assume that  $F = \{p_1, q_1, q_2, p_2\}$ , by the proof Lemma 33. Recall that  $\|p_1, C\| = 4$  and  $\|u', C\| = 0$  for some  $u' \in F$ . Furthermore, since  $\|R \setminus F, C\| = 0$ ,  $\|r, C\| = 0$ .

Suppose  $R \in \{C_5, K_{2,3}\}$ . Let  $F' := \{q_1, r, q_2, p_2\}$ , so that  $u' \in F'$ ,  $\|F', C\| \leq 8$  and  $\|F', R\| \leq 10$ . Since  $q_1q_2, rp_2 \notin E(G)$ ,  $\|F', C - C\| \geq 12(k - 2) + 2$  so that  $k \geq 3$  and  $\|F', C'\| \geq 13$  for some  $C' \in \mathcal{C} - C$ , a contradiction to Lemma 14.

Thus  $p_1r, p_2r \in E(G)$ . Since three of the five vertices in  $R$  send four edges to  $C$ , there exists  $i \in [2]$ , such that at least two vertices in  $\{r, q_i, p_i\}$  have four neighbors in  $C$ , and so have a common neighbor  $c \in C$ . This implies that  $G[\{r, q_i, p_i, c\}]$  contains a chorded cycle. Furthermore, there exists  $v \in \{p_{3-i}, q_{3-i}\}$  such that  $v$  has four neighbors in  $C$ , and so  $C - c + v$  contains a chorded cycle, a contradiction.

Thus,  $|C| \neq 4$  and  $G[C] \cong K_{3,3}$ , as desired.  $\square$

**structure**

**Lemma 36.** *If  $R \not\cong K_{2,3}$ , then  $G[C] \cong K_{3,3}$  for all  $C \in \mathcal{C}$ . If  $R \cong K_{2,3}$ , then  $G[C] \cong K_{3,3}$  for all but at most one  $C \in \mathcal{C}$ , and for any such  $C$ ,  $G[C] \cong K_{1,1,2}$  and  $G[V(R) \cup V(C)] \cong K_{1,4,4}$ .*

*Proof.* Let  $F \subseteq V(R)$  be as guaranteed by Lemma 33. If  $R$  is not isomorphic to  $K_{2,3}$ , then  $\|F, C\| \geq 12(k - 1)$ . By Lemma 14,  $\|F, C\| \leq 12$  for all  $C \in \mathcal{C}$  so that in fact, equality holds for all  $C \in \mathcal{C}$ . Thus, by Lemma 35,  $G[C] \cong K_{3,3}$  for all  $C \in \mathcal{C}$ .

So assume  $R \cong K_{2,3}$  with partite sets  $A = \{p_1, p_2, p_3\}$  and  $B = \{q_1, q_2\}$  with  $|A| = 3$  and  $|B| = 2$ . Since  $A$  and  $B$  are independent, we have  $\|B, C\| \geq 6k - 8$  and

$$2\|A, C\| = \sum_{a \in A} 2\|a, C\| \geq 3(6k - 2) - 12 = 18k - 18,$$

so  $\|A, C\| \geq 9(k - 1)$  and  $\|R, C\| \geq 15k - 17 = 15(k - 1) - 2$ . If  $\|R, C\| \geq 16$  for some  $C \in \mathcal{C}$ , then there exists some  $u \in R$  such that  $\|u, C\| = 4$ . By Lemma 14,  $\|R - u, C\| \leq 12$  so that there exists  $u' \in R - u$  such that  $\|u', C\| \leq 3$ . However,  $\|R - u', C\| \geq 13$ , a contradiction to Lemma 14.

We therefore have that, for ever  $C \in \mathcal{C}$ ,  $13 \leq \|R, C\| \leq 15$ . Fix  $C \in \mathcal{C}$ . At least two vertices in  $R$  have three neighbors each in  $C$  so that by Lemmas 13 and 15,  $|C| = 4$  or  $G[C] \cong K_{3,3}$ . We claim that  $G[C] \not\cong K_4$ .

Suppose on the contrary,  $G[C] \cong K_4$ . If  $\|p_i, C\| \geq 3$  for some  $i \in [3]$ , Lemma 14 implies that  $\|R, C\| \leq 12$ , a contradiction. So  $\|p_i, C\| \leq 2$  for all  $i \in [3]$ . Hence  $\|B, C\| \geq 7$  so that for all  $c \in C$  and  $j \in [2]$ ,  $C - c + q_j$  is a chorded cycle. As  $\|R, C\| \geq 13$ , there exists  $c \in C$  such that  $\|c, R\| \geq 4$ . Without loss of generality,  $N_R(c) \supseteq \{p_1, p_2, q_1\}$ . However,  $C - c + q_2$  and  $p_1cp_2q_1p_1$  each contain chorded cycles, a contradiction.

So for all  $C \in \mathcal{C}$ , either  $|C| = 4$  and  $C$  is singly chorded or  $G[C] \cong K_{3,3}$ . By Lemma 13,  $\|u, C\| \leq 3$  for all  $u \in A$  and  $C \in \mathcal{C}$ . Since  $\|A, C\| \geq 9(k-1)$ , we deduce that  $\|A, C\| = 9$  and so  $\|u, C\| = 3$  for all  $u \in A$  and  $C \in \mathcal{C}$ .

Suppose  $|C| = 4$  and  $C$  is singly chorded. We can label  $V(C) = \{c_1, c_2, c_3, c_4\}$  such that  $c_1c_2c_3c_4$  is a cycle and  $c_2c_4$  is the chord. By Lemma 13,  $uc_1, uc_3 \in E(G)$  for all  $u \in A$ . Since,  $C - c_i + u$  is a chorded cycle for  $i \in \{1, 3\}$ ,  $R - u + c_i$  cannot contain a chorded cycle, which implies that  $N_R(c_i) = A$ . Hence, for every  $v \in B$ ,  $N_C(v) \subseteq \{c_2, c_4\}$ , and since  $\|R, C\| \geq 13$ , equality holds and  $N_C(v) = \{c_2, c_4\}$  for every  $v \in B$ .

Fix  $u \in A$ . Without loss of generality, assume  $N_C(u) = \{c_1, c_3, c_4\}$ . Then  $C - c_2 + u$  is a chorded cycle. If  $u' \in A - u$  has  $c_2 \in N_C(u')$ , then  $R - u + c_2$  contains a chorded cycle, a contradiction. Thus, for all  $w \in A$ ,  $N_C(w) = \{c_1, c_3, c_4\}$  so that  $N_R(c_4) = V(R)$  and  $G[R \cup C] \cong K_{4,4,1}$ .

Recall that  $\|R, C\| \geq 15(k-1) - 2$  and  $\|R, C'\| \leq 15$  for all  $C' \in \mathcal{C}$ . Further,  $\|u, C'\| \leq 3$  for all  $u \in R$  and  $C' \in \mathcal{C}$ . Since  $\|R, C\| = 13$ ,  $\|R, C''\| = 15$  for every  $C'' \in \mathcal{C} - C$ . However, for any  $u \in A$ ,  $\|u, C''\| \leq 3$  so that  $F := R - u$  satisfies  $\|F, C''\| \geq 12$ . Furthermore,  $F$  satisfies all the hypotheses of Lemmas 33 and 35, so that  $G[C''] \cong K_{3,3}$  for all  $C'' \in \mathcal{C} - C$ .

This completes the proof of the lemma.  $\square$

pqchords

**Lemma 37.** *For every  $u \in R$  and  $C \in \mathcal{C}$ ,  $\|u, C\| \leq 3$ . If  $P'$  is path that spans  $R$ ,  $p$  is an endpoint of  $P'$  and  $q$  is adjacent to  $p$  on  $P'$ , then  $d_G(p) = 3k-1$  and  $d_G(q) \geq 3k-1$ . In particular, for every  $C \in \mathcal{C}$   $\|p, C\| = 3$  and  $\|q, C\| \geq 2$ .*

*Proof.* Let  $p$  and  $p'$  be the two endpoints of  $P'$ , and let  $q$  and  $q'$  be the neighbors of  $p$  and  $p'$ , respectively, on  $P'$ . By Lemmas 13 and 36,  $\|u, C\| \leq 3$  for all  $u \in R$  and  $C \in \mathcal{C}$ . Therefore, if  $d_R(u) = 2$ , then  $d_G(u) \leq 3k-1$ , so in particular,  $d_G(p) \leq 3k-1$  and  $d_G(p') \leq 3k-1$ . If  $pp' \notin E$ , then  $d_G(p') = d_G(p) = 3k-1$ . Otherwise,  $pp' \in E$  and  $p$  is not adjacent to  $q'$ . In this case,  $d_R(q') = 2$  so that  $d_G(p) = 3k-1$ . Since  $\|u, C\| \leq 3$  for all  $u \in R$  and  $C \in \mathcal{C}$ , it follows that  $\|p, C\| = 3$ . By symmetry, this holds for  $p'$  as well.

Since  $\|q, R\| \leq 3$ , if we can show that  $d_G(q) \geq 3k-1$ , it follows that  $\|q, C\| \geq 2$  for all  $C \in \mathcal{C}$ . So assume  $d_G(q) \leq 3k-2$ . Now,  $qp' \in E(G)$ , as otherwise  $d_G(q) \geq 3k-1$ . If  $|R| = 4$ , then by Lemma 31,  $R$  contains a chorded cycle. So  $|R| > 4$ , and as a result  $qq' \notin E(G)$ . Since  $d_G(q) \leq 3k-2$ , we get  $d_G(q') \geq 3k$ , and furthermore, since  $d_R(q') \leq 3$  and  $\|q', C\| \leq 3$  for all  $C \in \mathcal{C}$ , we deduce that  $\|q', C\| = 3$  and  $d_R(q') = 3$ . This implies  $pq' \in E(G)$ , as otherwise we get a chorded cycle in  $R$ . Furthermore,  $d_G(q) = 3k-2$  and  $\|q, R\| \leq 3$  so that  $\|q, C\| \geq 1$  for all  $C \in \mathcal{C}$ .

Since  $|R| \geq 5$ , there exists  $r' \notin \{p, p'\}$  a neighbor of  $q'$  on  $P'$ . Note that  $r' \in \mathcal{P}$  so that by the above,  $d_G(r') = 3k-1$  and  $\|r', C\| = 3$  for all  $C \in \mathcal{C}$ . If  $|R| \geq 6$ , then  $r'q \notin E(G)$  and  $d_G(q) \geq 3k-1$ , a contradiction. Hence,  $|R| = 5$ , and, furthermore,  $R \cong K_{2,3}$  with partite sets  $\{q, q'\}$  and  $\{p, p', r'\}$ . Observe that for all  $u \in \{p, r', q', p'\}$  and  $C \in \mathcal{C}$ ,  $\|u, C\| = 3$ .

If now fix  $C \in \mathcal{C}$ , such that  $\|q, C\| \leq 2$ , which must exist because  $d(q) = 3k-2$  and  $d_R(q) = 3$ . By Lemma 36,  $G[C] \in \{K_{3,3}, K_{1,1,2}\}$ . Furthermore, if  $G[C] \cong K_{1,1,2}$ , then  $G[C \cup R] = K_{1,4,4}$ , but this contradicts the fact that  $\|q', C \cup R\| = 6$ . Hence,  $C \cong K_{3,3}$  and let  $A$  and  $B$  denote its partite sets. By Lemmas 13 and 16, we may assume  $N_C(p) = N_C(r') = N_C(p') = A$ ,  $N_C(q') = B$ , and  $N_C(q) \subseteq B$ . Since  $\|q, C\| \leq 2$ , there exists  $b \in B \setminus N_C(q)$ . We can replace  $C$  with  $C - b + p'$  and replace  $P'$  with  $bq'P'p$ . Our new collection and path satisfy (O1)-(O3). However,  $b$  is an endpoint of our new path and by the above,  $d_G(b) = 3k-1$ . Since  $bq \notin E(G)$ ,  $d_G(q) \geq 3k-1$ , a contradiction.  $\square$

K32\_K22

**Lemma 38.**  *$R$  is either isomorphic to  $K_{2,3}$  or  $K_{2,2}$ .*

*Proof.* If  $|R| = 4$ , then Lemmas 31 implies that  $R \cong K_{2,2}$ , so assume  $|R| \geq 5$  and  $R$  is not isomorphic to  $K_{2,3}$ . Let  $P = u_1, \dots, u_{|R|}$ ,  $p := u_1$ ,  $q := u_2$ ,  $q' := u_{|R|-1}$  and  $p' := u_{|R|}$ . Let  $C \in \mathcal{C}$ . By Lemma 36,  $G[C] \cong K_{3,3}$ , so we let  $A = \{a_1, a_2, a_3\}$  and  $B = \{b_1, b_2, b_3\}$  be its partite sets. Recall that by Lemma 13, if  $\|u, C\| = 3$  for any  $u \in R$ , then  $N_C(u) \in \{A, B\}$ .

First assume that  $R$  is Hamiltonian (that is,  $R$  contains a cycle of size  $|R|$ ). Since every vertex in  $R$  is the endpoint of a path spanning  $R$ , by Lemma 37,  $\|u, C\| = 3$  for every  $C \in \mathcal{C}$  and  $u \in R$ . By Lemma 16, we can assume that  $N_C(u_i) = A$  if  $i$  is odd and  $N_C(u_i) = B$  if  $i$  is even. Therefore, Lemma 16 implies that  $|R|$  is even, which further implies that  $|R| \geq 6$ . Then for any  $a \in A$  and  $b \in B$ ,  $G[\{u_1, \dots, u_4, a, b\}]$  and  $C - a - b + u_5 + u_6$  contain chorded cycles, a contradiction.

So we can assume  $R$  is not Hamiltonian. Let  $pw$  be a hop on  $P$  so that  $w \neq p'$ . First assume  $w \neq q'$ . Without loss generality assume that  $N_C(p') = A$ . By Lemmas 16 and 37,  $N_C(p) \cap N_C(q) = \emptyset$ , and so there exists  $cc' \in E(C)$  such that  $pcc'qPwp$  is a cycle with chord  $pq$ . By Lemmas 16 and 37,  $|N_C(p') - c - c'| \geq 2$  and  $|N_C(q') - c - c'| \geq 1$ , so  $C - c - c' + p' + q'$  contains a chorded cycle, a contradiction.

Now we can assume that both  $pq'$  and  $qp'$  are edges. Since  $R \neq K_{2,3}$ , we have that  $|R| \geq 6$ . Let  $r \neq p$  and  $r' \neq p'$  be the neighbors of  $q$  and  $q'$ , respectively, on  $P$ . Note that  $r$  and  $r'$  are endpoints of paths spanning  $R$  so that  $\|r, C\| = \|r', C\| = 3$ . By Lemmas 16 and 37, and because  $pq', qp' \in E(G)$ , we may assume that  $N_C(p) = N_C(r) = N_C(r') = N_C(p') = A$  and  $N_C(q) \cup N_C(q') \subseteq B$ . In particular, we may assume  $qb_1 \in E(G)$  so that  $pa_1b_2a_2b_1qp$  is a cycle with chord  $pa_2$ , and  $rPp'a_3r$  is a cycle with chord  $a_3r'$ , a contradiction.

So  $|R| = 5$  and  $R \cong K_{2,3}$ , as desired.  $\square$

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**Lemma 39.** *If  $G[C] \cong K_{3,3}$  for every  $C \in \mathcal{C}$ , then  $G \cong G_1(n, k)$ .*

*Proof.* By Lemma 38,  $R \in \{K_{2,2}, K_{2,3}\}$ . So let  $U_1, U_2 \subseteq V(R)$  be the partite sets of  $R$  such that  $|U_1| \geq |U_2| = 2$ , and let  $u_1 \in U_1$ ,  $V_2 := N_G(u_1)$ , and  $V_1 := V(G) \setminus V_2$ . Since  $u_1$  is the end of spanning path of  $R$ , Lemma 37 implies that  $|V_2| = 3k - 1$ . Since  $|G| \leq 6(k - 1) + 5$ ,  $|V_1| \leq 3k$ . We aim to show that  $N_G(v) = V_2$  for all  $v \in V_1$ . This will imply that  $G \cong G_1(n, k)$ .

Fix  $v \in V_1 - u_1$ . Since  $u_1v \notin E(G)$ , Lemma 37 implies that  $d_G(v) \geq 3k - 1$ . If  $v \in U_1$ , then  $v$  is the end of a spanning path of  $R$ , and by Lemmas 13, 16 and 37,  $N_G(v) = N_G(u_1) = V_2$ . So we may assume  $v \in V_1 \setminus U_1$ , and in particular,  $v \in C$  for some  $C \in \mathcal{C}$ .

Define  $V'_1 := \{u \in V_1 : \|u, U_2\| \geq 1\}$ , and suppose  $v \in V'_1 \setminus U_1$ . Recall that we are assuming  $G[C] \cong K_{3,3}$  for all  $C \in \mathcal{C}$  so that by Lemma 13,  $G[C - v + u_1] \cong K_{3,3}$ . Furthermore,  $v$  is an end of a path of length  $|R|$  in  $R' := R - u_1 + v$ . This new collection and path satisfy (O1)-(O3), so by Lemma 38,  $R' \cong R$  and  $N_G(v) = N_G(u_1) = V_2$ .

Now suppose  $v \in V_1 \setminus V'_1$ . Since  $d_G(v) \geq 3k - 1$  and  $v$  has at most  $3(k - 1)$  neighbors in  $V_2$ ,  $v$  must have two neighbors in  $V_1$ . By Lemmas 16 and 37, for every  $u_2 \in U_2$ ,  $d_G(u_2) \geq 3k - 1$  and  $N_G(u_2) \subseteq V_1$ , so that  $|V'_1| \geq 3k - 1$ . Since  $|V_1| \leq 3k$ ,  $v$  has a neighbor, say  $v'$ , in  $V'_1$ . However, by the above,  $N_G(v') = V_2$ , which contradicts the fact that  $vv'$  is an edge. Therefore,  $V'_1 = V_1$  which finishes the proof of the lemma.  $\square$

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**Lemma 40.** *Suppose there exists  $C \in \mathcal{C}$  with  $|C| = 4$ . Then  $G \cong G_2(k)$ .*

*Proof.* By Lemmas 36 and 38, we can assume  $R \cong K_{2,3}$ ,  $G[C] \cong K_{1,1,2}$ , and  $G[R \cup C] \cong K_{1,4,4}$ . Let  $A'$  and  $B'$  be the two partite sets of size four and  $\{c\}$  be the partite set of size one in  $G[R \cup C]$ . By symmetry, we can assume that any  $v \in A' \cup B'$  is an end of a spanning path in  $R$  or the end of a spanning path of  $G[V(G) \setminus V(C')]$  for some collection  $\mathcal{C}'$  of  $k - 1$  vertex disjoint cycles that satisfies (O1)-(O3), so, by Lemma 37,  $d_G(v) = 3k - 1$  and  $\|v, C - C'\| = 3(k - 2)$ . By Lemma 36, for all  $D \in \mathcal{C} - C$ ,  $G[D] \cong K_{3,3}$ , and, with Lemma 16, we deduce that  $\|v, D\| = 3$  and that we can label the partite sets of  $D$  as  $A_D$  and  $B_D$  so that for every  $p \in A'$ ,  $N_D(p) = B_D$  and for every  $q \in B'$ ,  $N_D(q) = A_D$ . Therefore, there exists a partition  $\{A, B, \{c\}\}$  of  $V(G)$  such that for every  $p \in A'$ ,  $N_G(p) = B + c$ , for every  $q \in B'$ ,  $N_G(q) = A + c$ , and  $|A| = |B| = 3k - 2$ .

If  $u \in V(G) \setminus (A' \cup B')$ , then there exists  $D \in \mathcal{C} - C$ , such that  $u \in D$ . Let  $p \in A' \cap V(R)$ , and  $q \in B' \cap V(R)$  and label  $\{w, w'\} = \{p, q\}$  so that  $uw \notin E(G)$  and  $uw' \in E(G)$ . We have that  $G[D - u + w] \cong K_{3,3}$  and  $G[R - w + u] \cong K_{3,2}$ , so there exists a collection  $\mathcal{C}'$  of  $k - 1$  vertex disjoint cycles containing  $C$  that satisfies (O1)-(O3), and there exists a spanning path of  $G[V(G) \setminus V(C')]$  such that  $u$  is an endpoint or  $u$  is the neighbor of an endpoint. Therefore, by Lemma 37,  $d_G(u) \geq 3k - 1$ , so, with Lemma 36, we have that  $N_G(u) = (V(C) \setminus N_C(w')) + c$  and, for any  $D' \in \mathcal{C}' - C$ , by Lemma 16,  $N_{D'}(u) = D' \setminus N_{D'}(w')$ . Therefore, either  $N_G(u) \supseteq B + c$  if  $u \in A$  or  $N_G(u) \supseteq A + c$  if  $u \in B$ . Hence,  $G$  contains  $G_2(k)$  as a spanning subgraph. As  $G_2(k)$  is edge-maximal with respect to not containing  $k$  disjoint chorded cycles,  $G \cong G_2(k)$ .  $\square$

Using Lemmas 36, 38, 39, and 40, we conclude  $G \in \{G_1(n, k), G_2(k)\}$ .

## 5 Concluding Remarks

remarks

Many variations on Theorems 1 and 5 have appeared, and suggest further extensions of Theorem 9. We present only a small selection below.

A result of Gould, Hirohata, and Horn [8] implies the following:

**Theorem 41.** *Let  $G$  be a graph on  $|G| \geq 6k$  vertices with  $\delta(G) \geq 3k$ . Then  $G$  contains  $k$  disjoint doubly chorded cycles.*

While it is not clear that  $|G| \geq 6k$  is necessary, it would be interesting to characterize the sharpness examples for this theorem; that is, if  $|G| \geq 6k$  and  $\delta(G) = 3k - 1$  but  $G$  does not contain  $k$  disjoint doubly chorded cycles, what does  $G$  look like? For more results on the existence of  $k$  disjoint multiply chorded cycles, see [9]

Additionally, rather than consider  $\delta(G)$  or  $\sigma_2(G)$ , one may consider the neighborhood union,  $\min\{|N(x) \cup N(y)| : xy \in E(\overline{G})\}$ . See the following results.

**Theorem 42** (Faudree-Gould, [6]). *If  $G$  has  $n \geq 3k$  vertices and  $|N(x) \cup N(y)| \geq 3k$  for all nonadjacent pairs of vertices  $x, y$ , then  $G$  contains  $k$  disjoint cycles.*

**Theorem 43** (Gould-Hirohata-Horn, [8]). *Let  $G$  be a graph on at least  $4k$  vertices such that for any nonadjacent  $x, y \in V(G)$ ,  $|N(x) \cup N(y)| \geq 4k + 1$ . Then  $G$  contains  $k$  disjoint chorded cycles.*

**Theorem 44** (Gould-Hirohata-Horn, [8]). *Let  $G$  be a graph on  $n > 30k$  vertices such that for any nonadjacent  $x, y \in V(G)$ ,  $|N(x) \cup N(y)| \geq 2k + 1$ . Then  $G$  contains  $k$  disjoint cycles.*

**Theorem 45** (Qiao, [13]). *Let  $r, s$  be nonnegative integers, and let  $G$  be a graph on at least  $3r + 4s$  vertices such that for any nonadjacent  $x, y \in V(G)$ ,  $|N(x) \cup N(y)| \geq 3r + 4s + 1$ . Then  $G$  contains  $r + s$  disjoint cycles,  $s$  of them chorded.*

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## References

- [BFG] [1] A. Bialostocki, D. Finkel, and A. Gyárfás, Disjoint chorded cycles in graphs, *Discrete Mathematics* 308 (2008), no. 23, 5886-5890.
- [CFGL] [2] S. Chiba, S. Fujita, Y. Gao, and G. Li, On a sharp degree sum condition for disjoint chorded cycles in graphs. *Graphs Combin.* 26 (2010), no. 2, 173-186.
- [CH] [3] K. Corrádi and A. Hajnal, On the maximal number of independent circuits in a graph, *Acta Math. Acad. Sci. Hungar.* 14 (1963) 423-439.
- [Di] [4] G. Dirac, Some results concerning the structure of graphs, *Canad. Math. Bull.* 6 (1963) 183-210.
- [E] [5] H. Enomoto, On the existence of disjoint cycles in a graph, *Combinatorica* 18(4) (1998) 487-492.
- [FG] [6] J. Faudree and R. Gould, A note on neighborhood unions and independent cycles, *Ars Combin.* 76 (2005), 29-31. 05C69

- [F] [7] D. Finkel, On the number of independent chorded cycles in a graph, *Discrete Mathematics* 308 (2008) no 22, 5265-5268.
- [GHH] [8] R. Gould, K. Hirohata, and P. Horn, Independent cycles and chorded cycles in graphs, *J. Comb.* 4 (2013), no. 1, 105122.
- [GHM] [9] R. Gould, P. Horn, and C. Magnant, Multiply chorded cycles, *SIAM J. Discrete Math.* 28 (2014), no. 1, 160-172.
- [KKY] [10] H. A. Kierstead, A. V. Kostochka, and E. C. Yeager, On the Corrádi-Hajnal Theorem and a question of Dirac, *Journal of Combinatorial Theory, Series B*, to appear.
- [KKY2] [11] H. A. Kierstead, A. V. Kostochka, and E. C. Yeager, The  $(2k - 1)$ -connected multigraphs with at most  $k - 1$  disjoint cycles, *Combinatorica*, to appear.
- [KKMY] [12] H. A. Kierstead, A. V. Kostochka, T.N. Molla, and E. C. Yeager, Sharpening an Ore-type version of the Corrádi-Hajnal Theorem, *submitted*.
- [Q] [13] S. Qiao, Neighborhood unions and disjoint chorded cycles in graphs, *Discrete Mathematics* 312 (2012), no. 5, 891-897.
- [W] [14] H. Wang, On the maximum number of disjoint cycles in a graph, *Discrete Mathematics* 205 (1999) 183-190.